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The Multiple Production of Mesons

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In this paper an attempt is made to treat the impacts of nucleons of very high energy ($\gg Mc^2$). These impacts lead to meson emission, despite the relatively small momentum transfers to be expected, because changes of nuclear charge and spin require a readjustment of the nuclear meson fields. Where the fields are strong (as e.g., in the pseudoscalar case), meson emission is multiple, and the multiplicity increases with energy. For weaker coupling (e.g., scalar fields) this is not true.

Two closely related methods are developed for a more quantitative description of these collisions. These methods may be expected to give valid results when the collision time is short compared to the periods of the emitted mesons, and when the emission of mesons into the various modes can be treated as statistically independent. Under these conditions the total cross section may be expected to be about 10^{-26} cm^2, and the multiplicity to approach for high energies $2g^{\frac{1}{2}}(ME_0/2\pi\mu^2c^2)^{\frac{1}{4}}$, with E_0 the primary energy and g the dimensionless coupling constant.

Some applications to the calculation of positive excess, primary spectrum, angular distribution, and the theory of auger showers, are discussed briefly in the final section.

I. QUALITATIVE CONSIDERATIONS

YUKAWA'S original argument, which anticipated the existence of mesons, was based on the role which they would play in accounting for nuclear stability and the forces between nucleons. On the one hand, very extensive theoretical study has failed to devise a theory which accounts at all reasonably for the quantitative aspects of nuclear forces; on the other hand, decisive experience with cosmic-ray mesons, both with the absorption and scattering of the penetrating component, and with the capture of mesons brought to rest in matter, fails to reveal the evidences of any strong interaction. Only one phenomenon shows, if not yet conclusively, that there is a strong interaction between mesons and nuclear matter. This is the phenomenon of the production of mesons, which takes place with very high probability, and with characteristic and high multiplicity when the primary cosmic rays traverse the matter of the earth's atmosphere.[1] These present notes give some considerations about the theoretical treatment of these collisions.

Quite apart from the inadequacy of current theoretical methods and concepts, to which we will have to return over and over again in the discussion of this problem, it appears that a somewhat more complex model may have to be invoked to reconcile the strong interactions manifested in meson creation with the absence of subsequent interactions between cosmic-ray mesons and nuclear matter. Especially if one compares the processes of production with the processes of meson absorption, let us say, in the domain of ten billion volts, one is led to an apparent lack of correspondence between elementary processes and their approximate inverses quite at variance with preceding experience. In the discussions of this problem, which has definitely not been resolved by such theoretical development as the "strong coupling" theory, it was recognized at the Shelter Island Conference[2] that the most natural interpretation lay in the existence of "structure," either for the nucleons or the mesons, or both. Thus Weisskopf suggested the existence of long-lived metastable states for the nucleons formed during the primary collision and subsequently decaying with a long lifetime to yield the mesons; whereas Marshak proposed the equally satisfactory view that the mesons originally created were metastable with regard to disintegration into those actually observed, as might, for instance, be the case if they had a higher mass.

In making this study, we shall, of course, not

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[1] M. Schein, W. P. Jesse, and E. O. Wollan, Phys. Rev. **59**, 615 (1941); M. Schein, M. Iona, and J. Tabin, Phys. Rev. **64**, 253 (1943).

[2] June 1947, sponsored by the National Academy of Sciences.

be able to account for copious meson production without assuming a strong coupling between mesons and nucleons; but it is to be understood that much of the argument does not depend on a direct identification of the mesons with those observed in cosmic rays, nor on the exclusion of metastable states for nuclear matter. In fact, the arguments which originally led Yukawa to the view that charged quanta must exist have in no way been weakened by recent experience in the domain of nuclear physics: the scattering, by neutrons and protons, of neutrons of high energy reveals the important part played by charge exchange in these phenomena; and the measurement of the magnetic moment of the triton strongly suggests, if it does not prove, the existence of electromagnetic currents associated with nuclear attractions.[3]

Even under the assumption of strong coupling, such as, for instance, characterized the symmetric pseudoscalar or "mixed" theories, or the pair theories, of nuclear forces, an understanding of the phenomena of meson production is by no means trivial. There seem to be two characteristics of these collisions which are well established: The first is that they correspond to a cross section very close to the size of the nucleus; the second is that the typical act is a multiple one in which a considerable number of mesons, perhaps 5 or so, for typical cosmic-ray energies, are created.[4] When one remembers that these collisions occur at distances of encounter for which nuclear forces as normally experienced are extremely weak, one must ask two questions: Why do these encounters lead to such strong emission, and why is this emission characteristically multiple?

To these two questions one would like to give simple qualitative answers. To the first: although relatively remote collisions are not accompanied by large momentum transfers between nucleons, they probably are accompanied by changes of spin and isotopic spin. Thus, the collision produces a discrepancy between the meson fields which are actually present about the nucleon, and those which are appropriate to its new state. Insofar as it is possible to idealize the collision processes as rapid, compared to the time of emission of mesons, this would lead us to expect a very probable emission. As to the second question,[5] and guided by similar qualitative arguments, we may suppose that if the fields of nucleons are indeed very intense, so that large numbers of mesons are present in these fields, then the effect of a sudden change of spin of isotopic spin will be to release considerable numbers of mesons, as is observed. Whether and how this number will depend on the energy of the collision must, as the future analysis will show, depend importantly on detailed characteristics of the field. But even at this stage we may say that if the collisions between nucleons involved no change other than a small transfer of momentum, the large radiation cross sections would not be intelligible, and that if the fields surrounding the nucleons were in fact as weak and as weakly dependent on position as the Coulomb field, this radiation process would not be highly multiple.

It should be observed that in these discussions we have assumed that the primary cosmic ray is itself a nucleon, an assumption which is now current. Some support for the view that meson production is in fact the result of the collision of nucleons comes from the study of so-called penetrating showers throughout the atmosphere, and at sea level.[6] In fact, the characteristics and number of such showers at sea level, and the equal distribution between charged and neutral initiators, suggests quite strongly that these phenomena are what is left of the primary collisions in the upper atmosphere, after the absorption of the primaries and of their secondary protons and neutrons. At least this assumption that protons initiate these collisions in the upper atmosphere underlies the work of this paper.

Most of the evidence of multiple production and of collision cross section comes from the typical cosmic ray, of energy say ten billion volts. It is possible so to interpret the very large air showers as to obtain evidence bearing on much higher energies. It seems to be true that these showers have their origin quite near the top of

[3] Felix Villars, Phys. Rev. **72**, 256 (1947).
[4] Cf. reference 1.

[5] J. R. Oppenheimer and J. Schwinger, Phys. Rev. **60**, 150 (1941); H. A. Bethe, Phys. Rev. **70**, 787 (1946).
[6] L. Jańossy, Proc. Roy. Soc. **A179**, 361 (1942); L. Jańossy and G. D. Rochester, Proc. Roy. Soc. **A181**, 399 (1943); W. B. Fretter, Bull. Am. Phys. Soc. Stanford meeting, July 1947.

the atmosphere, and that they involve an intimate mixture of meson and soft radiation. A natural, but by no means unique, interpretation of this is that the soft radiation arises primarily from the decay of neutral mesons, though little in our arguments would be altered were the decay process to involve transitions from heavier to lighter particles, provided the time of such decay were short enough to satisfy the Mills-Christy condition,[7] and long compared to all collision times. If these suggestions prove to be correct, they will bring with them the consequence that, even for energies in the range of the 10^{14} volts, and perhaps higher, collision cross sections must be of the order of nuclear areas, and must lead to the conversion of a large part of the primary energy into mesons and their decay products. As we shall see, the interpretation of such collisions on the basis of meson-field theories gives us information about field strengths of very great magnitude, such as might obtain in the immediate neighborhood of the nucleon.

It is, of course, not possible, even adopting one of the more or less standard forms of meson theory, to compute what will happen in a collision between two nucleons of high energy, because of this clearly depends on a reliable theory of the reaction of meson radiation on the collision process. Even in the simple case of the electromagnetic radiation of a slow electron deflected by a scatterer no such theory exists; nor do even the most radical attempts at altering electrodynamics give an answer which is finite and acceptable.[8] But precisely for this reason, attempts at treating meson radiation may not be entirely devoid of methodological interest, since effects of radiation reaction, which in electrodynamics may be small and only barely susceptible to observation, are here of decisive importance.

For analogy, we may be guided by the problem of electromagnetic radiation. Here, as Bloch and Nordsieck first showed, the problem can be very simply treated insofar as one may make three assumptions: (1) the collision time is short compared to the periods of the emitted radiation; (2) the effect of the emitted radiation, in particular its recoil, on the source may be neglected; (3) only those components of the radiation field will be considered for which these conditions are satisfied. The difficulties, of course, arise from the fact that in any radiation problem, the "virtual" effects of components for which conditions 1 and 2 are not satisfied cannot legitimately be ignored. When these three conditions are fulfilled, one has two simple results: The radiated spectrum can be obtained as the difference between the quasi-static fields of the radiator as it is and as it should be after the scattering process, and (2) the total probability of collision is not altered, though collisions are now accompanied by the emission of radiation, which would have been regarded as elastic had not the coupling with the radiation field been considered.

In the following we shall attempt to develop methods based on the three assumptions above, and appropriate to meson-producing collisions. To this end, we shall first modify the Bloch-Nordsieck method of canonical transformation to make it apply to characteristic meson fields. As is well known, this means physically regarding the nucleon, plus its quasi-static field, as the elementary ingredient of the collision rather than the bare nucleon itself. Although it is easy to carry through these calculations, they may nevertheless leave something to be desired, and that for the following reason: in nuclear impacts and for the emission of mesons, the spin and isotopic spin of the nucleons change. On the one hand, these changes cannot be consistently taken into account by a theory which neglects the effects of radiation on the source. On the other hand, just these changes in spin would appear to be decisive for the course of the collision. For this reason we have supplemented the Bloch-Nordsieck treatment by a species of perturbation theory, which in the electromagnetic case would itself be its equivalent, and which again exploits, but to a more limited extent, the statistical independence of the emission process. In this treatment we considered the process of emitting a fixed number of mesons, and calculated its

[7] M. M. Mills and R. F. Christy, Phys. Rev. **71**, 275 (1947).

[8] W. Pauli, Helv. Phys. Acta **19**, 234 (1946); H. A. Bethe and J. R. Oppenheimer, Phys. Rev. **70**, 451 (1946). *Note added in proof:* Recently, in connection with the work of Lamb, Schwinger, Bethe and Weisskopf on the term shifts of hydrogen, one of us (H. W. L.) has solved this problem (Physical Review; to be published). The extension of this work to meson problems clearly deserves intensive study. It is to be hoped that it makes many of the assumptions of this paper unnecessary.

cross section to the lowest order in all coupling constants in which it appears in the formal treatment. We then compare the cross sections for various numbers of mesons emitted, always working in their lowest order. We *assume* that although these absolute cross sections are manifestly meaningless, their ratios are correctly given by this means of calculation. In fact, this assumption means that all calculated cross sections will be reduced by the effects of radiation damping, so that, as in the electromagnetic case, the sum of all cross sections is the same as we would obtain for the elastic cross section with the neglect of radiation processes and, further, that the factor by which the calculated cross section must be reduced varies not at all, or only slowly, with the number of mesons emitted. It is clear that quite apart from deeper questions, some consideration needs to be given to the momentum relations in these collisions, since collisions involving large transfers of momentum from one nucleon to another can take place only for correspondingly small impact parameters. Some comments on this point will be found in the course of the actual calculations.

Thus the calculations which follow make no pretense to an estimate of total cross sections, but take these over from the estimate of the probability of the corresponding elastic collisions. It may be agreed that on the basis of what is known about nuclear forces and of the measurement on the collisions of 100-million volt neutrons with protons and neutrons,[9] the cross section observed for the cosmic-ray processes would seem to be a reasonable extrapolation. In fact, if one assumes that there are no serious singularities in the potentials between neutrons and protons, and adjusts their range and strength to agree with nuclear experience, one obtains elastic cross sections in the relativistic domain which are constant, and have the value of about 10^{-26} cm^2 per nucleon. This is in reasonable enough accord with cosmic-ray experience.

One final comment may perhaps be appropriate. These calculations have been carried out in the relativistic domain, that is, where the primary energies are large compared to a billion volts; this has been done in part in an attempt to make the results applicable to the great auger showers, and this in turn in part because this application of theory extends to the most intense fields and the smallest distances. Since it will be clear from the following calculations that in theories like the scalar theory, where these fields are only moderate, one does not obtain a high and increasing multiplicity of meson production, the question arises as to what meaning to ascribe to the enormously greater field strengths characteristic, e.g., of the pseudoscalar, or pair, theories. For certainly we know one thing: the nucleon response to these fields gives no direct evidence of their existence; the nucleons act as though the forces between them were free of important singularities, not only in the normal phenomena of nuclear physics, but even by the collisions of 100-million volt nucleons, where quite small momentum exchanges are the rule. Nevertheless, the possibility should not be overlooked that these strong fields may have meaning, in the sense that they can be radiated under suitable circumstances, and yet not be susceptible of familiar interpretation in terms of the behavior of the nucleons which are subject to them. Much will have to be learned before even the partial validity of this notion can be regarded as established.

In summary of the results of the calculations which have just briefly been described, it may be said that a scalar theory gives an energy independent, and, in general, a small multiplicity of production; that a theory which involves no internal change in the nucleons when they scatter each other, such as charge or spin exchange, will give a very small probability of radiation; that the symmetrical pseudoscalar theory, and a similar formulation of pair theory, do in fact give large probabilities and large multiplicity of meson production; and that in the forms here considered, and at extremely high energies, the multiplicity rises with the cube root of the primary energy. Some physical applications of these results are discussed in a quite tentative way in Section IV of the paper.

II. BLOCH-NORDSIECK METHOD

A. Symmetrical Scalar Theory

We will now treat the collision of two nucleons in the symmetrical scalar theory by a method

[9] We are indebted to Dr. E. MacMillan for information on n–p and n–n *total* cross sections at about 90 Mev.

similar to that of Bloch and Nordsieck. The meson field will be considered to interact with the nucleon through the latter's isotopic spin, which will be treated classically. A canonical transformation will be performed to find the quasi-static meson field associated with the nucleon for a definite orientation of its isotopic spin (definite charge state). We will describe charge exchange in the collision by introducing a potential, V, which can change the isotopic spin of the nucleons in the collision. The corresponding change in the quasi-static field appropriate to the isotopic spin orientation will give us the radiation in the collision. Thus the primary interaction between nucleons could, for instance, be of the form $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)V(r)$; of V we shall assume that its Fourier transform tends to zero for momenta $\gg\mu c$. As will be obvious, we should obtain no radiation (in the approximation here considered) were we to omit the charge exchange $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ from the interaction. It should further be observed that here we neither assume nor deny that the interaction can be derived from the meson interactions themselves. Following Bloch and Nordsieck,[10] we use the Dirac equation for the heavy particle coupled to the meson field, and neglect recoils. The Hamiltonian is then,*

$$H=\boldsymbol{\alpha}\cdot\mathbf{P}+\beta M+\sum_{\alpha k}\frac{k_0}{2}(Q_{k\alpha}{}^2+P_{k\alpha}{}^2)+\beta g(4\pi)^{\frac{1}{2}}\sum_{\alpha k}\frac{\tau_\alpha}{k_0^{\frac{1}{2}}}(Q_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r}+P_{k\alpha}\cos\mathbf{k}\cdot\mathbf{r}) \quad (1)$$

which, by the Bloch-Nordsieck argument,** can be replaced by its positive energy part,

$$H=\mathbf{v}\cdot\mathbf{P}+\frac{M}{\gamma}+\sum_{\alpha k}\frac{k_0}{2}(Q_{k\alpha}{}^2+P_{n\alpha}{}^2)+\frac{g}{\gamma}(4\pi)^{\frac{1}{2}}\sum_{\alpha k}\frac{\tau_\alpha}{k_0^{\frac{1}{2}}}(Q_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r}+P_{k\alpha}\cos\mathbf{k}\cdot\mathbf{r}), \quad (2)$$

where the isotopic spin vector $\boldsymbol{\tau}$ is treated as a classical unit vector. M is the mass of the heavy particle, and the meson field is represented by

$$\varphi_\alpha=\sum_k k_0^{-\frac{1}{2}}(Q_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r}+P_{k\alpha}\cos\mathbf{k}\cdot\mathbf{r}), \quad (3)$$

where the $Q_{k\alpha}$ and $P_{k\alpha}$ are canonically conjugate.

We make the canonical transformation

$$\begin{aligned}P_{k\alpha}&=P_{k\alpha}'-a_{k\alpha}\cos\mathbf{k}\cdot\mathbf{r}',\\ Q_{k\alpha}&=Q_{k\alpha}'-a_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r}',\\ \mathbf{r}&=\mathbf{r}',\\ \mathbf{P}&=\mathbf{P}'+\sum_{\alpha k}\mathbf{k}a_{k\alpha}(Q_{k\alpha}'\sin\mathbf{k}\cdot\mathbf{r}'+P_{k\alpha}'\cos\mathbf{k}\cdot\mathbf{r}'-\tfrac{1}{2}a_{k\alpha}),\end{aligned} \quad (4)$$

with

$$a_{k\alpha}=\frac{g\tau_\alpha(4\pi/k_0)^{\frac{1}{2}}}{\gamma(k_0-\mathbf{v}\cdot\mathbf{k})},$$

obtaining finally, for the transformed wave equation,

$$\left\{\mathbf{v}\cdot\mathbf{P}'+\frac{M}{\gamma}+\sum_{\alpha k}\frac{k_0}{2}(Q_{k\alpha}{}^2+P_{k\alpha}{}^2)-\sum_k\frac{2\pi g^2}{\gamma^2(k_0-\mathbf{v}\cdot\mathbf{k})}-E\right\}u'(\mathbf{r}',Q_{k\alpha}')=0, \quad (5)$$

whose solution is

$$u'=\chi\exp[iM\gamma\mathbf{v}\cdot\mathbf{r}']\prod_{\alpha k}h_{m_{\alpha k}}(Q_{k\alpha}'), \quad (6)$$

where χ is a normalized spinor, and the $h_{m_{\alpha k}}$ are normalized Hermite polynomials.

Now, transforming back to the old coordinates, we have

$$u(\mathbf{r},Q_{k\alpha})=\chi\exp[iM\gamma\mathbf{v}\cdot\mathbf{r}]\prod_{\alpha k}\exp[-ia_{k\alpha}\cos\mathbf{k}\cdot\mathbf{r}(Q_{k\alpha}+\tfrac{1}{2}a_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r})\times h_{m_{\alpha k}}(Q_{k\alpha}+a_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r}). \quad (7)$$

The cross section for scattering by a potential V, with emission of $n_{\alpha k}$ mesons in each mode $\alpha\mathbf{k}$, is

$$d\sigma=\frac{P_0Q_0Q}{P}\frac{d\Omega}{4\pi^2}|V_{PQ}|^2\prod_{\alpha k}\frac{w_{\alpha k}{}^{n_{\alpha k}}}{n_{\alpha k}!}e^{-w_{\alpha k}}, \quad (8)$$

where $(\mathbf{P},P_0)$ and $(\mathbf{Q},Q_0)$ are the initial and final momentum and energy of the nucleon, and

$$w_{\alpha k}=\frac{2\pi g^2(\Delta\tau_\alpha)^2}{\gamma^2k_0(k_0-\mathbf{v}\cdot\mathbf{k})^2}, \quad (9)$$

where $\Delta\tau_\alpha$ is the change in τ_α during the scattering process.

It should be observed that the term $e^{-w_{\alpha k}}$ above is independent of the $n_{\alpha k}$ and should be summed over all the modes available. This would

[10] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

* $c=\hbar=\mu=$ mass of the meson $=1$.

** It will be seen in Section III that virtual pair production does not play an important part in these processes. In (1), α and β are the usual Dirac matrices, $Q_{k\alpha}$ is the amplitude of the mode of the meson field with wave number $\mathbf{k}$ and charge coordinate α, $P_{k\alpha}$ is the momentum conjugate to $Q_{k\alpha}$, k_0 is $[1+k^2]^{\frac{1}{2}}$, and g^2 is the dimensionless coupling constant, γM is the energy of the nucleon.

make the total cross section zero, corresponding to the fact that the virtual emissions are not restricted by the energy conservation conditions. A similar result was found by Pauli and Fierz[11] for the electromagnetic case. However, the fact that this factor is independent of the $n_{\alpha k}$ means that its effect is to cut down the total cross section, but it does not affect the *ratio* of the probabilities of emitting N and $N+1$ mesons. We will *assume* that the inadequacy of the assumptions inherent in this method, as discussed in Section I, does not destroy this characteristic of the damping. In this way we can give sense to the calculation, since we will calculate only the most probable number of mesons emitted, and will attach no significance to the total cross section. Therefore we omit these terms, and all others that do not depend upon the number of emitted mesons. We also replace $n_{\alpha k}!$ by 1, since the finite rest mass makes it unlikely that more than one meson will be emitted into any particular mode. We have, then,

$$\sigma \sim |V_{PQ}|^2 \prod_{\alpha k} w_{\alpha k}, \tag{10}$$

where the product is carried out over all the modes into which a meson is emitted.

Now to obtain the total cross section for the emission of N mesons, with total energy loss ϵ, this has to be summed over all the modes satisfying the energy loss condition. Thus

$$\sigma(N,\epsilon)d\epsilon \sim |V_{PQ}|^2 \left(\frac{g^2}{4\pi^2\gamma^2}\right)^N \sum \int \cdots \times \int \prod_n \frac{d\mathbf{k}_n}{k_{0n}(k_{0n}-\mathbf{v}\cdot\mathbf{k}_n)^2} \prod_\alpha \frac{(\Delta\tau_\alpha^2)^{N_\alpha}}{N_\alpha!}, \tag{11}$$

where the sum is taken over all N_α such that $\sum N_\alpha = N$, and the integral over the volume in momentum space corresponding to an energy loss between ϵ and $\epsilon+d\epsilon$.*** The $N_\alpha!$ arise from the indistinguishability of mesons of each type. We have, then,

$$\sigma(N,\epsilon)d\epsilon \sim |V_{PQ}|^2 \left(\frac{g^2}{2\pi\gamma^2}\right)^N \frac{(\Delta\boldsymbol{\tau}^2)^N}{N!} \times \int \cdots \int \prod_n \frac{k_n dk_{0n} d\mu_n}{(k_{0n}-v\mu_n k_n)^2}, \tag{12}$$

where μ_n is the cosine of the angle the nth meson makes with the direction of motion of the nucleon. Now, in order to find the limits of integration, it is necessary to take into account the longitudinal momentum transfer from the other nucleon (for which exactly these same considerations hold, provided we are in a coordinate system in which each nucleon is near the speed of light).

If the momentum transfer between the nucleons is $\mathbf{K}$, then the equations of conservation of energy and momentum are

$$\begin{aligned} \Delta P_{1z} &= -\sum^{N_1} k_z - K_z, \\ \Delta P_{2z} &= -\sum^{N_2} k_z + K_z, \\ \Delta E_1 &= -\epsilon_1 = \mathbf{v}_1\cdot\Delta\mathbf{P}_1 = -v_1 \sum^{N_1} k_z - v_1 K_z, \\ \Delta E_2 &= -\epsilon_2 = \mathbf{v}_2\cdot\Delta\mathbf{P}_2 = v_2 \sum^{N_2} k_z - v_2 K_z, \end{aligned} \tag{13}$$

where the z axis is taken along the direction of motion of the first nucleon. Thus, since

$$\begin{aligned} &\sum^N k_{0n} = \epsilon_1+\epsilon_2, \\ &\epsilon_1 = \frac{v_1}{v_1+v_2} \sum^N (k_{0n}+v_2 k_{zn}) = \sum^N \alpha_n, \\ &\epsilon_2 = \frac{v_2}{v_1+v_2} \sum^N (k_{0n}-v_1 k_{zn}) = \sum^N \beta_n. \end{aligned} \tag{14}$$

Now making the transformation from μ_n, $k0_n$ to α_n, β_n we have, since

$$kdk_0 d\mu = \frac{v_1+v_2}{v_1 v_2} d\alpha d\beta \sim 2 d\alpha d\beta,$$

$$\sigma(N_1 N_2 \epsilon_1\epsilon_2)d\epsilon_1 d\epsilon_2 \sim |V(K_z)| \left(\frac{g^2}{4\pi}\right)^{N_1+N_2} \times \frac{(\Delta\boldsymbol{\tau}_1^2)^{N_1}(\Delta\boldsymbol{\tau}_2^2)^{N_2}}{\gamma_1^{2N_1}\gamma_2^{2N_2} N_1! N_2!} \int \cdots \int \prod_n \frac{d\alpha_n d\beta_n}{(\alpha_n^2)^{N_2}(\beta_n^2)^{N_1}}, \tag{15}$$

where in the denominator α^2 appears for each meson emitted by the second nucleon, and β^2 for each emitted by the first. The integration is over the sheet

$$\epsilon_1 \leqslant \sum^N \alpha_n \leqslant \epsilon_1 + d\epsilon_1, \quad \epsilon_2 \leqslant \sum^N \beta_n \leqslant \epsilon_2 + d\epsilon_2. \tag{16}$$

For ϵ_1 and ϵ_2 large compared to μc^2, the integral

[11] W. Pauli and M. Fierz, Nuovo Cimento 15, 167 (1938).

*** It should be noted that the continuous decrease in γ affects only the absolute magnitude of the cross section, so that the initial value may be used here.

in (15) becomes

$$\int \cdots \int \prod^{N_1} \left(\frac{4\gamma_1^2 \alpha_n d\alpha_n}{\alpha_n^2+\gamma_1^2} \right) \cdot \prod^{N_2} \left(\frac{4\gamma_2^2 \beta_n d\beta_n}{\beta_n^2+\gamma_2^2} \right),$$

since for a given α the lower limit of β is $(\alpha^2+\gamma_1^2)/4\alpha\gamma_1^2$, and inversely. This can be approximately evaluated when $\epsilon_1/N_1\gamma_1$ and $\epsilon_2/N_2\gamma_2$ are either much less than or much greater than one. We shall see that the most probable multiplicity and energy losses occur in the latter region, so we will evaluate the integral in that case. The major contributions will then come from $\alpha_n \sim \epsilon_1/N_1$, $\beta_n \sim \epsilon_2/N_2$, and the integral becomes, in this limit,

$$4^N \gamma_1^{2N_1} \gamma_2^{2N_2} \left(\ln \frac{\epsilon_1}{\gamma_1} \right)^{N_1-1} \left(\ln \frac{\epsilon_2}{\gamma_2} \right)^{N_2-1} \frac{N_1 N_2}{\epsilon_1 \epsilon_2} d\epsilon_1 d\epsilon_2.$$

Thus

$$\sigma(N_1 N_2 \epsilon_1 \epsilon_2) d\epsilon_1 d\epsilon_2 \sim |V(K_z)|^2 \times \left(\frac{g^2}{\pi} \right)^N \frac{N_1 N_2}{\epsilon_1 \epsilon_2} (\Delta \boldsymbol{\tau}_1^2)^{N_1} (\Delta \boldsymbol{\tau}_2^2)^{N_2} \times \frac{(\ln \epsilon_1/\gamma_1)^{N_1-1}}{N_1!} \frac{(\ln \epsilon_2/\gamma_2)^{N_2-1}}{N_2!} d\epsilon_1 d\epsilon_2. \quad (17)$$

Now to account for the different distributions of the mesons among the nucleons, this has to be multiplied by the binomial coefficient $C_{N_1}{}^N$, and summed over N_1 to obtain the cross section for the emission of N mesons, with energy losses ϵ_1 and ϵ_2.

$$\sigma(N\epsilon_1\epsilon_2) d\epsilon_1 d\epsilon_2 \sim \left(\frac{g^2}{\pi} \right)^N N! \sum_{N_1=0}^{N} |V(K_z)|^2 \times \frac{N_1 N_2}{\epsilon_1 \epsilon_2} (\Delta \boldsymbol{\tau}_1^2)^{N_1} (\Delta \boldsymbol{\tau}_2^2)^{N_2} \times \frac{(\ln \epsilon_1/\gamma_1)^{N_1-1}}{N_1!^2} \frac{(\ln \epsilon_2/\gamma_2)^{N_2-1}}{N_2!^2} d\epsilon_1 d\epsilon_2. \quad (18)$$

The $|V|^2$ is placed inside the sum, since, from (13), setting $v_1 \sim v_2 \sim c$,

$$2K_z = \epsilon_1 + \epsilon_2 + \sum^{N_2} k_z - \sum^{N_1} k_z = \epsilon_1 + \epsilon_2 - \sum^{N} k_{\|n}, \quad (19)$$

so that the longitudinal momentum transfer required depends upon the distribution of the mesons between the nucleons. We can obtain an idea of what is required by using the average values of the k_{zn} as given by the distribution (11). Here the resonance denominators force the emissions to be forward for the most part, and K_z is of the order $N\mu c$. Now our assumption that both nucleons are moving near the speed of light requires us, for large energy losses, to work in a coordinate system in which neither nucleon reverses its direction of motion. For large equal energy losses this is the center of mass system. We therefore† go into the center of mass system, setting $\gamma_1 = \gamma_2 = \gamma$, and suppose that $N \ll \gamma$. This corresponds to the condition that the mesons emitted have energies greater than Mc^2. The Fourier analysis of the scattering potential should be flat up to momentum transfers of order $\gamma\mu c$, so that the $|V|^2$ can now be taken out of the sum.

Integrating over-all energy losses, we obtain

$$\sigma(N) \sim \left(\frac{g^2}{\pi} \right)^N N! \left(\ln \frac{\epsilon_{\max}}{\gamma} \right)^N \sum_{N_1=0}^{N} \frac{(\Delta \boldsymbol{\tau}_1^2)^{N_1} (\Delta \boldsymbol{\tau}_2^2)^{N_2}}{N_1! N_2!}, \quad (20)$$

where $\epsilon_{\max}$ is the maximum energy a nucleon can lose in the center of mass system. This probably is somewhere between $(\gamma-1)M$ and $(\gamma-2)M$, since the nucleon does not convert its rest energy into mesons, and probably stops appreciable emissions when its velocity becomes sensibly less than that of light. Thus this factor becomes $\ln M/\mu$ for high energies, greater than around 10^{11} volts.

Now for a symmetrical potential of the form $(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) V(r)$ as is given by the symmetrical meson theories, $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$ is a constant of the motion in a scattering process, so $\Delta \boldsymbol{\tau}_1 = -\Delta \boldsymbol{\tau}_2 = \Delta \boldsymbol{\tau}$, and, since the sum in (20) may be replaced to sufficient accuracy by its largest term, we have,

$$\sigma(N) \sim \left(\frac{4g^2 \Delta \boldsymbol{\tau}^2}{\pi} \right)^N \frac{1}{N!} \left(\ln \frac{\epsilon_{\max}}{\gamma} \right)^N, \quad (21)$$

since $(\frac{1}{2}N)! \sim 2^{-N/2}(N!)^{\frac{1}{2}}$. This is valid for $N < \epsilon_{\max}/\gamma$. The cross section has a maximum at

$$\bar{N} = \frac{4g^2 \Delta \boldsymbol{\tau}^2}{\pi} \ln \frac{\epsilon_{\max}}{\gamma}, \quad (22)$$

which is small for most reasonable values of g^2, and gives a most probable multiplicity which

† Thus, if E_0 is the primary energy,
$\gamma M = [\frac{1}{2}M(E^0+M)]^{\frac{1}{2}}$.

does not increase with energy.†† This corresponds to the fact that the scalar meson field has only a r^{-1} singularity near the source, and contains only a logarithmically infinite number of mesons.

B. Symmetrical Pseudoscalar Theory

We will now use essentially the same method to treat the case of a symmetrical pseudoscalar field, which is different from the scalar theory in two respects. First, we will use a coupling in which both the isotopic and ordinary spins appear, so that changes in either one can cause meson emission. Second, we will use a gradient coupling so that the quasi-static field has a worse singularity near the source than was the case for the scalar field ($1/r^2$ instead of $1/r$). As we will see, this latter leads to a more copious meson emission, corresponding to the much stronger meson-field strengths in the vicinity of the source. We will suppose that the internucleonic potential can effect either ordinary or isotopic spin changes, but that it is free of singularities—and its Fourier transform small for $k \gg \mu c$—as would not be the case were the interaction to be derived directly from the meson fields; such singular interaction can clearly not be treated as small. The procedure is then exactly analogous to that for the scalar field, with the Namiltonian

$$H=\boldsymbol{\alpha}\cdot\mathbf{P}+\beta M+\sum_{\alpha k}\frac{k_0}{2}(Q_{k\alpha}{}^2+P_{k\alpha}{}^2)+g(4\pi)^{\frac{1}{2}}\sum_{\alpha k}\frac{\tau_\alpha}{k_0^{\frac{1}{2}}}(\boldsymbol{\sigma}\cdot\mathbf{k}-\gamma_5 k_0)\times(Q_{k\alpha}\cos\mathbf{k}\cdot\mathbf{r}-P_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r}), \quad (23)$$

which can be replaced by

$$H=\mathbf{v}\cdot\mathbf{P}+\frac{M}{\gamma}+\sum_{\alpha k}\frac{k_0}{2}(Q_{k\alpha}{}^2+P_{k\alpha}{}^2)-g(4\pi)^{\frac{1}{2}}\sum_{\alpha k}\frac{\tau_\alpha\sigma_{||}}{k_0^{\frac{1}{2}}}(k_0-k_{||})\times(Q_{k\alpha}\cos\mathbf{k}\cdot\mathbf{r}-P_{k\alpha}\sin\mathbf{k}\cdot\mathbf{r}) \quad (24)$$

†† Were we to assume very large values of g^2, we should have the other limiting case $N \gg \epsilon_{\max}/\gamma$; here (21) would be replaced by

$$\sigma(N)\sim\left(\frac{4g^2M^2\Delta\boldsymbol{\tau}^2}{\pi}\right)^N\left(\frac{\epsilon_{\max}}{\gamma M}\right)^{2N}\frac{1}{N!^3}, \quad (21a)$$

which yields

$$\bar{N}\sim(4g^2\Delta\boldsymbol{\tau}^2\epsilon_{\max}{}^2/\pi\gamma^2)^{\frac{1}{4}},$$

which, though large for large enough g^2, still does not increase with increasing energy.

for $v\sim c$. The canonical transformation is similar to the scalar one, and we obtain again

$$\sigma\sim|V_{PQ}|^2\prod v_{\alpha k}, \quad (25)$$

where now

$$v_{\alpha k}=(2\pi g^2/k_0)[\Delta(\tau_\alpha\sigma_{||})]^2, \quad (26)$$

so that

$$\sigma(N_1N_2\epsilon_1\epsilon_2)d\epsilon_1d\epsilon_2\sim|V(K_z)|^2\left(\frac{g^2}{\pi}\right)^N\frac{q_1{}^{N_1}q_2{}^{N_2}}{N_1!N_2!}\times\int\cdots\int\prod^N d\alpha_n d\beta_n, \quad (27)$$

where

$$q=\sum_\alpha[\Delta(\tau_\alpha\sigma_{||})]^2.$$

The integration yields,

$$\sigma(N_1N_2\epsilon_1\epsilon_2)d\epsilon_1d\epsilon_2\sim|V(K_z)|^2\left(\frac{g^2}{\pi}\right)^N\frac{q_1{}^{N_1}q_2{}^{N_2}}{N_1!N_2!}\times N^2\frac{(\epsilon_1\epsilon_2)^{N-1}}{N!^2}d\epsilon_1d\epsilon_2. \quad (28)$$

Again multiplying by $C_{N_1}{}^N$ and summing, we have

$$\sigma(N\epsilon_1\epsilon_2)d\epsilon_1d\epsilon_2\sim\left(\frac{g^2}{\pi}\right)^N\frac{N^2(\epsilon_1\epsilon_2)^{N-1}}{N!}d\epsilon_1d\epsilon_2\sum_{N_1=0}^N|V(K_z)|^2\frac{q_1{}^{N_1}q_2{}^{N_2}}{(N_1!N_2!)^2}. \quad (29)$$

Now, however, for large energy losses in the center of mass system the spherical symmetry gives

$$K_z\sim\tfrac{1}{2}(\epsilon_1+\epsilon_2),$$

which can be as large as $\epsilon_{\max}$, which is larger than $\gamma\mu c$ for incident proton energies greater than about 5×10^{10} ev. However, the spherical symmetry also makes the momentum transfer independent of the distribution of the mesons between the two nucleons, so that the matrix element can be taken out of the summation, and we will return to the question of the large momentum transfers below.

Replacing the sum, as before, by its largest term, which occurs at $N_1=f_1N$ and $N_2=f_2N$, where $f_1=q_1/(q_1+q_2)$ and $f_2=q_2/(q_1+q_2)$, we have

$$\sigma(N\epsilon_1\epsilon_2)d\epsilon_1d\epsilon_2\sim\left(\frac{g^2}{\pi}\right)^N\frac{N^2(\epsilon_1\epsilon_2)^{N-1}}{N!^3}\left(\frac{q_1+q_2}{f_1{}^{f_1}f_2{}^{f_2}}\right)^N d\epsilon_1d\epsilon_2, \quad (30)$$

and, integrating over-all energy losses in the center of mass system,

$$\sigma(N)\sim\left(\frac{g^2}{\pi}\right)^N \frac{\epsilon_{\max}{}^{2N}}{N!^3}\left(\frac{q_1+q_2}{f_1{}^{f_1}f_2{}^{f_2}}\right)^N, \qquad (31)$$

(31) has a maximum at

$$\bar{N}=\left(\frac{g^2\gamma^2M^2}{\pi}\right)^{\frac{1}{3}}\left(\frac{q_1+q_2}{f_1{}^{f_1}f_2{}^{f_2}}\right)^{\frac{1}{3}}\left(\frac{\epsilon_{\max}}{\gamma M}\right)^{\frac{2}{3}} \qquad (32)$$

which increases as $\gamma^{\frac{2}{3}}$ or as the cube root of the primary energy. This corresponds to the inverse square singularity of the pseudoscalar meson field, which gives an unlimited supply of mesons. We have assumed that the $|V(K_s)|^2$ varies more slowly with N than the other factors, even in the region of large momentum transfers.

We note, for future reference, that the case of a neutral pseudoscalar field is obtained by setting $f_1=f_2=\frac{1}{2}$, and $q_1=q_2=[\Delta(\tau_z\sigma_{||})]^2$, for the case of a symmetrical interaction between the nucleons. Thus, the cross section becomes,

$$\sigma(N)=\left(\frac{4g^2}{\pi}\right)^N \frac{\epsilon_{\max}{}^{2N}}{(N!)^3}[\Delta(\tau_z\sigma_{||})]^{2N}$$

and gives

$$\bar{N}=\left(\frac{4g^2\gamma^2M^2}{\pi}\right)^{\frac{1}{3}}[\Delta(\tau_z\sigma_{||})]^{\frac{2}{3}}\left(\frac{\epsilon_{\max}}{\gamma M}\right)^{\frac{2}{3}}. \qquad (32a)$$

We note also that the number of mesons of each charge type depends upon the change in $\tau_\alpha\sigma_{||}$ so that with a potential that doesn't discriminate among the different τ_α we should expect that an average of one-third of the mesons would be neutral, and two-thirds charged.

C. Comparison and Total Cross Section

The preceding considerations have illustrated a number of general features of the radiation process. First, it is clear from the method, as well as from the examples, that the radiation occurs through a change in the source—in particular, a change in some feature of the source to which the meson field is coupled. Thus the scalar field can be radiated by a change in the isotopic spin of the source, and the pseudoscalar by a change in either the ordinary or isotopic spin.

Secondly, the number of mesons emitted is largely determined by the number previously existing in the virtual field. Therefore, in order to obtain an appreciably energy-dependent multiplicity of production, we must have a badly singular meson field (worse than $1/r$), and cannot cut off the singularity except at very short distances. This makes the self-energy and nuclear forces difficulties acute, and a quantitative discussion of the cut-off question will be deferred to Section IV.

Another difference between the scalar and pseudoscalar theories is in the angular distribution of the emitted mesons. For the scalar case the average perpendicular momentum of an emitted meson is of order μc, whereas for the pseudoscalar mesons the distribution is spherically symmetric in the center of mass system. This has the consequence that the emission of scalar mesons requires only a small momentum exchange between the nucleons, and the absolute cross section is not limited by this. Thus we might suppose that the total cross section for inelastic scattering is of the same order as that for elastic scattering, around $(\hbar/\mu c)^2$, which is in rough accord with the data on the absorption of primaries at the top of the atmosphere.

On the other hand, the pseudosclar mesons require a momentum exchange between the nucleons of the order γMc in the center of mass system, and, if the potential between them is similar to a Lorentz-contracted Yukawa well, this means an impact parameter $\hbar/Mc$. However, this extreme situation obtains only for energies greater than around 10-in. volts. For lower energies the factor $\epsilon_{\max}/\gamma M$ becomes important, and the important impact parameters become of order $\hbar c\gamma/\epsilon_{\max}$ which increases the cross section somewhat. Thus, at 10^{10} volts primary energy, $\epsilon_{\max}$ is perhaps 3 to $5\mu c^2$ and $\hbar c\gamma/\epsilon_{\max}\sim\frac{1}{2}\hbar/\mu c$, which is not too small.

The evaluation of the physical consequences of these considerations is not unambiguous. For they show that only for close impact ($<\hbar/Mc$ for extremely high energies) can probabilities of meson emission be regarded as statistically independent; and only to this case are the methods of this paper applicable. For the more distant collisions, in which large momentum transfers are not readily available, there are two possibilities: either the statistical independence upon which the methods of this paper are based will be maintained, in which case such distant collisions will not, in general, be accompanied by radiation. Or the requirement of small momentum transfers may force correlations among the successive emissions, in which case the methods here used are no longer applicable. However, one might schematize the correlations by supposing, for example, that the volume in momentum space available to the mesons is decreased by the momentum transfer requirement. In order, for instance, to cut down the momentum transfer from γMc to $\gamma\mu c$, a factor of roughly ten, the emissions might be concentrated forward, into about one-fifth of the previously available momentum space per meson. Thus a factor of $1/5^N$ might be introduced into Eq. (31), and $5^{-\frac{1}{3}}$ or 0.6 into Eq. (32).

Although present theoretical methods do not settle which alternative will be realized, the experimental evidence for auger showers shows that even at these energies the cross sections for converting primary into cascade radiation are still of the order of 1/100 barn per nucleon.

We also note that 'cut-off' procedures in the meson fields would tend to concentrate the emissions forward, level off the multiplicity as a function of energy, and decrease the required momentum transfer between the nucleons.

We may also note some more general features of the multiplicity laws. The fact that the differential cross section must be a relativistic invariant, for all numbers of mesons emitted, means, in the case of statistical independence of the successive emissions, that that part of the cross section which involves the meson emission must be a product of invariant factors. Thus, in the scalar case we have

$$\prod \frac{d\mathbf{k}_n}{k_{0n}\gamma^2(k_{0n}-\mathbf{v}\cdot\mathbf{k}_n)^2},$$

and in the pseudoscalar $\prod d\mathbf{k}_n/k_{0n}$. In the case where the emissions are statistically independent, (and only in this case) the cross section will break up into a product. Since only the two four-vectors ($\mathbf{k}$, k_0) and ($\mathbf{P}$, E) can appear, the cross section must then have the form

$$\prod \frac{d\mathbf{k}_n}{k_{0n}} f[\gamma(k_{0n}-\mathbf{v}\cdot\mathbf{k}_n)].$$

The two cases studies so far correspond to $f(x)$ a simple power, $f(x)=1/x^2$ for the scalar theory, and $f(x)=1$ for the pseudoscalar, For a general case $f(x)=x^\nu$, some of the properties of the solution are immediately clear. First, for ν negative the emissions are concentrated in a forward direction, increasingly so as ν becomes more negative. For $\nu=0$, we have spherically symmetric emissions in the center of mass system, and for $\nu>0$ the mesons tend to be concentrated backward somewhat. One can do the integrations over the momentum space roughly, and finds that, for $\nu \geqslant 0$, $\sigma(N)\sim\gamma^{(2\nu+2)N}/N!^{3+\nu}$, apart from numerical factors. The case of the pseudoscalar field corresponds to $\nu=0$.

If ν is less than zero, the cross section behaves as const.$/N!^{3+\nu}$ for $N\ll M/\mu$, and const.$/N!^{1-\nu}$ for $N\gg M/\mu$, so that no negative values of ν give a multiplicity which varies appreciably with energy. The scalar field corresponds to $\nu=-2$. As ν increases, the multiplicity will also tend to increase more rapidly with energy.

III. PERTURBATION METHOD

A. Neutral Pseudoscalar

Now, in order to take into account the quantum properties of the ordinary and isotopic spin in a more consistent manner, we will treat the case of a neutral pseudoscalar field by a sort of perturbation theory,[12] giving an explicit form to the spin and isotopic spin dependence of the potential function. In the electromagnetic case this would be the formal equivalent of the Bloch-Nordsieck method. We will use the Born approximation, in the sense that each process will be considered only in the first order in which it occurs, and we will use a potential function proportional to $(\boldsymbol{\tau}\cdot\mathbf{T})(\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}-\gamma_5\Gamma_5)$, which has the property of flipping either the spin or isotopic spin of the nucleons. Here, as later, the small Greek letters refer to the first nucleon, and the large ones to the second. The matrix elements for the emission of a meson by the first or second

[12] We are indebted to Dr. H. A. Bethe for helpful methodological discussions during the early part of this work.

nucleon, respectively, are

$$g\left(\frac{2\pi}{k_0}\right)^{\frac{1}{2}} \tau_z\sigma_{||}(k_0-k_z),$$
$$g\left(\frac{2\pi}{k_0}\right)^{\frac{1}{2}} T_z\Sigma_{||}(k_0+k_z), \quad (33)$$

where, as before, the pair-producing parts have been omitted, and the nucleons have been supposed near the speed of light. We take only the even parts since the large energy denominators associated with pair production will cut down the contribution of the odd parts as $1/\gamma$ so that one can neglect them.

The matrix element for the emission of N_1 and N_2 mesons by the first and second nucleon, respectively, is

$$H=\frac{(2\pi g^2)^{N/2}\prod_n b_n}{\prod_n k_{0n}^{\frac{1}{2}}} \times\sum\frac{\chi_f{}^*\Lambda_f\sigma_{||}\tau_z\Sigma_{||}T_z\cdots V_{\text{scatt}}\sigma_{||}\tau_z\cdots\Lambda_i\chi_i}{\text{energy denominators}}, \quad (34)$$

where χ_i and χ_f are the initial and final state spinors (each for two nucleons), Λ_i and Λ_f are the corresponding projection operators, and b_n is $k_{0n}-k_{zn}$ for a meson emitted by the first nucleon, and $k_{0n}+k_{zn}$ for one emitted by the second nucleon. The energy denominators are of the form

$$b_1(b_1+b_2)\cdots(-b_{N-1}-b_N)(-b_N),$$

where the break occurs at the scattering act. If there are n_1 emissions by the first nucleon before the scattering, and n_2 by the second, then the sum is over n_1, n_2, the combinations of n_1 and n_2 mesons, and all the permutations of the emissions before and after the scattering. Since the $\sigma_{||}$, $\Sigma_{||}$, τ_z, and T_z all commute, the sum over permutations is done directly, and with

$$\sum_{\text{Perm}}\frac{1}{b_1(b_1+b_2)(b_1+b_2+b_3)\cdots}=\frac{1}{\prod_n b_n}$$

yields

$$H=\frac{(2\pi g^2)^{N/2}}{\prod_n k_{0n}^{\frac{1}{2}}}\sum[(-)^{n_1+n_2}\chi_f{}^*\Lambda_f(\sigma_{||}\tau_z)^{N_1-n_1} \times(\Sigma_{||}T_z)^{N_2-n_2}V_{\text{scatt}}(\sigma_{||}\tau_z)^{n_1}(\Sigma_{||}T_z)^{n_2}\Lambda_i\chi_i]. \quad (35)$$

Now the operator part of V_{scatt} can be broken up into a part that commutes with the $\sigma_{||}\tau_z$ and a part that anticommutes. Thus

$$(\boldsymbol{\tau}\cdot\mathbf{T})(\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}-\gamma_5\Gamma_5) =[\tau_zT_z(\sigma_z\Sigma_z-\gamma_5\Gamma_5)+(\boldsymbol{\tau}_\perp\cdot\mathbf{T}_\perp)(\boldsymbol{\sigma}_\perp\cdot\boldsymbol{\Sigma}_\perp)] +[\tau_zT_z(\boldsymbol{\sigma}_\perp\cdot\boldsymbol{\Sigma}_\perp)+(\boldsymbol{\tau}_\perp\cdot\mathbf{T}_\perp)\sigma_z\Sigma_z] \quad (36)$$

and, calling the former of these A^+ and the latter A^-, we have

$$H=V(K_z)\frac{(2\pi g^2)^{N/2}}{\prod_n k_{0n}^{\frac{1}{2}}}\{\chi_f{}^*\Lambda_f(\sigma_{||}\tau_z)^{N_1}(\Sigma_{||}T_z)^{N_2} \times\sum[(-)^{n_1+n_2}A^++A^-]\Lambda_i\chi_i\}, \quad (37)$$

where

$$\sum[(-)^{n_1+n_2}A^++A^-] =\sum_{n_1n_2}C_{n_1}{}^{N_1}C_{n_2}{}^{N_2}[(-)^{n_1+n_2}A^++A^-]=2^NA^-.$$

The loss of the A^+ means that only that part of the potential that can flip either the spin or isotopic spin of the nucleons is effective in causing radiation. We have then,

$$H=V(K_z)\frac{(8\pi g^2)^{N/2}}{\prod_n k_{0n}^{\frac{1}{2}}}[\chi_f{}^*\Lambda_f(\sigma_{||}\tau_z)^{N_1} \times(\Sigma_{||}T_z)^{N_2}A^-\Lambda_i\chi_i]. \quad (38)$$

Squaring, then averaging over the initial and summing over the final spins (ordinary and isotopic), we have

$$|H|^2=\tfrac{1}{16}|V(K_z)|^2\frac{(8\pi g^2)^N}{\prod_n k_{0n}}Sp[\Lambda_fA^-\Lambda_iA^-]. \quad (39)$$

The spur is non-vanishing, and independent of the N, so

$$|H|^2\sim|V(K_z)|^2\frac{(8\pi g^2)^N}{\prod_n k_{0n}}, \quad (40)$$

and

$$d\sigma\sim|V(K_z)|^2\left(\frac{g^2}{\pi^2}\right)^N\prod^N\frac{d\mathbf{k}_n}{k_{0n}}. \quad (41)$$

Now the integration over the momentum space of the mesons, and summation over N_1, can be carried out as in Section II, yielding

$$d\sigma\sim|V(K_z)|^2\left(\frac{8g^2}{\pi}\right)^N\frac{N^2(\epsilon_1\epsilon_2)^{N-1}}{N!^3}d\epsilon_1d\epsilon_2, \quad (42)$$

which, as before, favors large energy losses, and

$$\sigma(N) \sim \left(\frac{8g^2}{\pi}\right)^N \frac{\epsilon_{\max}{}^{2N}}{N!^3} \tag{43}$$

for the integrated cross section in the center of mass system, which has a maximum at

$$\bar{N} = \left(\frac{8g^2\gamma^2 M^2}{\pi}\right)^{\frac{1}{3}} \left(\frac{\epsilon_{\max}}{\gamma M}\right)^{\frac{2}{3}}. \tag{44}$$

Comparing (43) with the result for the neutral pseudoscalar field obtained by the Bloch-Nordsieck method in Section II, we see that the only difference is the replacement of $(\Delta\tau_z\sigma_{||})^2$ by 2, indicating that an exchange potential is in fact equivalent to a large spin change in the Bloch-Nordsieck method.

We could, of course, treat a neutral scalar field by exactly this same method, with similar results.

The extension of the above to the case of the symmetrical theory is not trivial, since the feature that made the combinatorics simple, namely, the commutativity of the τ_z, is not present when charged mesons appear. Thus the successive emissions are not independent, and the simplicity is lost. In this particular case the non-commutativity of the successive emissions is related to the conservation of charge. In our treatment by the Bloch-Nordsieck method we neglected this difficulty by treating the isotopic spin vector classically. Such classical treatment would correspond to an extreme form of the so-called strong coupling theories, implying in particular the existence of charge isobars of the nucleons. Thus the charge asymmetry of the emitted mesons is compensated for by the fact that the heavy particles emerge from the collision in isobar states, in such a way as to effect the over-all conservation of charge. This problem does not exist in a neutral theory.

B. A Pair Theory

We wish now, in view of the fact that the emission of single mesons may be almost forbidden, to consider a typical spinor-pair theory. We use an interaction that is sensitive to spin changes of the nucleons,

$$H_{\text{int}} = g\Psi^*\psi^*(\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma} - \gamma_5\Gamma_5)\psi\Psi, \tag{45}$$

where the capital letters refer to the heavy particle, and the small ones to the mesons. Taking the even part for the heavy particle, we can replace this by

$$H_{\text{int}} = g\Psi^*\Sigma_{||}\Psi\psi^*(\sigma_{||} - \gamma_5)\psi, \tag{46}$$

where we will later take the odd part of the light particle operators.

Writing down the matrix element for the emission of N *pairs* in an exactly analogous manner to that used in the preceding section, and with the same potential between the nucleons, we obtain

$$H = V(K_z)\frac{(2g)^N}{\prod_n b_n}[\chi_f{}^*\Lambda_f\Sigma_{1||}{}^{N_1}\Sigma_{2||}{}^{N_2}(\boldsymbol{\Sigma}_{1\perp}\cdot\boldsymbol{\Sigma}_{2\perp})\Lambda_i\chi_i]$$
$$\prod_n \psi_n{}^*\lambda_n{}^+(\sigma_{||} - \gamma_5)\lambda_n{}^-\psi_n, \tag{47}$$

where, as before, the Λ and λ are the projection operators for the heavy and light particles, respectively, and

$$b_n = \epsilon_n{}^+ + \epsilon_n{}^- - p_{n||}{}^+ - p_{n||}{}^-.$$

Squaring, then summing over final and averaging over initial spins, we have

$$|H|^2 = \frac{1}{4}|V(K_z)|^2\frac{(4g^2)^N}{\prod_n b_n{}^2}$$
$$\cdot Sp[\Lambda_f(\boldsymbol{\Sigma}_{1\perp}\cdot\boldsymbol{\Sigma}_{2\perp})\Lambda_i(\boldsymbol{\Sigma}_{1\perp}\cdot\boldsymbol{\Sigma}_{2\perp})]$$
$$\cdot\prod_n Sp[\lambda_n{}^+(\sigma_{||} - \gamma_5)\lambda_n{}^-(\sigma_{||} - \gamma_5)]. \tag{48}$$

The former spur is non-vanishing, and independent of N, and the latter, since the momentum of the "positron" is the negative momentum of the negative energy state, is

$$(2/\epsilon_n{}^+\epsilon_n{}^-)(\epsilon_n{}^+ - p_{n||}{}^+)(\epsilon_n{}^- - p_{n||}{}^-).$$

Therefore,

$$d\sigma \sim (8g^2)^N \prod_n \frac{d\mathbf{p}_n{}^+ d\mathbf{p}_n{}^-}{\epsilon_n{}^+\epsilon_n{}^-} \times \frac{(\epsilon_n{}^+ - p_{n||}{}^+)(\epsilon_n{}^- - p_{n||}{}^-)}{(\epsilon_n{}^+ + \epsilon_n{}^- - p_{n||}{}^+ - p_{n||}{}^-)^2}, \tag{49}$$

which has an invariant form of the sort discussed in Section II. Integration of this expression over the momentum space available, followed by summation over N_1, would give the total cross section for the emission of N pairs, but, unfortunately, the integration is not as simple as in the previous cases. However, to obtain an estimate of energy dependence, we may set the final factor in (49) equal to unity, and have

$$\sigma(N)\sim\epsilon_{\max}{}^{4N}/N!^2(2N)!^2,$$

which has a maximum for

$$\bar{N}\sim\epsilon_{\max}{}^{\frac{1}{2}}$$

which, by coincidence, is the same result as was found for the single pseudoscalar theory.

IV. SOME APPLICATIONS

A. Angular Distribution

We wish now to apply the foregoing results to the auger showers, and to the primary production of mesons. For this purpose we will use the results of the symmetrical pseudoscalar theory, as obtained in Section II. We found there that the weighting in momentum space was proportional to

$$\prod_n \frac{d\mathbf{k}_n}{k0_n}\propto\prod_n d\alpha_n d\beta_n,$$

with the α_n and β_n defined by (14). Averaging over all the mesons except one, we have for the average value of any function of α and β

$$\langle f(\alpha,\beta)\rangle_{\text{Av}}=\frac{(N-1)^2}{(\epsilon_1\epsilon_2)^{N-1}}\int_0^{\epsilon_2}\int_0^{\epsilon_1} f(\alpha,\beta)\times(\epsilon_1-\alpha)^{N-2}(\epsilon_2-\beta)^{N-2}d\alpha d\beta. \quad (50)$$

Thus the average energy of the emitted mesons is obtained by setting $f(\alpha,\beta)=\alpha+\beta$, and gives

$$\langle k_0\rangle_{\text{Av}}=(\epsilon_1+\epsilon_2)/N,$$

as it must.

Now approximating the distribution function for high energies, and transforming it to the laboratory system, we have a convenient form for applications,

$$P(\mathbf{k})d\mathbf{k}\cong\frac{N^2}{2\pi ME_0}\exp\left[-N\left(\frac{k_0}{E_0}+\frac{1}{2Mk_0}\right)\right]\times\exp[-Nk_0\vartheta^2/2M]\frac{d\mathbf{k}}{k_0}, \quad (51)$$

where, as before, energies are measured in units of μc^2 and masses in units of the meson mass.

In order to obtain the angular distribution alone, we have to integrate this over k_0, and obtain approximately, for small ϑ,

$$P(\vartheta)d\Omega\cong\frac{2M}{\pi E_0}\exp\left[-\frac{N^2}{4M^2}\left(\vartheta^2+\frac{2M}{E_0}\right)\right]\times\frac{d\Omega}{(\vartheta^2+2M/E_0)^2}. \quad (52)$$

This involves, however, an integration over meson momenta as low as $N/2M$, many of which will not reach sea level, and, in order to find the sea level angular distribution, we should only go down to energies of the order of 2×10^9 ev, which is the energy required to come through the atmosphere. With the lower limit $\eta=2\times10^9$ ev we find

$$P'(\vartheta)d\Omega\cong\frac{2M}{\pi E_0}\exp\left[-\frac{N\eta}{2M}\left(\vartheta^2+\frac{2M}{E_0}\right)\right]\times\frac{d\Omega}{(\vartheta^2+2M/E_0)^2}, \quad (53)$$

which has two characteristic widths

$$\vartheta_1\sim(2M/E_0)^{\frac{1}{2}},\quad \vartheta_2\sim(2M/N\eta)^{\frac{1}{2}}, \quad (54)$$

and we must choose whichever is smaller. Estimating the coefficients in (32), we find $\bar{N}\sim2E_0^{\frac{1}{3}}$ so that for the most probable multiplicity,

$$\vartheta_1\sim4E_0^{-\frac{1}{2}},\quad \vartheta_2\sim E_0^{-1/6}, \quad (55)$$

and for $E_0>5\times10^{10}$ ev, ϑ_1 is smaller, and we have roughly

$$P''(\vartheta)d\Omega\sim\frac{2M}{\pi E_0}\frac{d\Omega}{(\vartheta^2+2M/E_0)^2} \quad (56)$$

for angles less than ϑ_2.

This corresponds to spherically symmetric emissions in the center of mass system, and corresponds to an extremely wide distribution for the most highly populated primary energies. In the region of the auger showers, where $E_0\sim10^{14}$ or 10^{15} ev, the half-width is around 4×10^{-3} radian. This is, of course, the spread of the hard core of the showers, and the initial spread of the neutral mesons that form the soft part. The latter is later considerably spread by multiple scattering.

If the showers are created at altitudes of (say) 20 km, this angle corresponds to a diameter of approximately 80 meters at sea level.

B. Primary Spectrum

The fact that the average multiplicity of production increases with the energy of the primary proton causes the energy spectrum of the mesons to be different from that for the primary protons. In particular, if the primary spectrum is proportional to $E_0^{-\lambda}dE_0$ and the average multiplicity is proportional to E_0^r, then the average energy ϵ of an emitted meson is E_0^{1-r} and the meson spectrum at the place of production is proportional to

$$\epsilon^{-\lambda/(1-r)}\cdot\epsilon^{1/(1-r)-1}d\epsilon\cdot\epsilon^{r/(1-r)} \quad \text{or} \quad \epsilon^{-(\lambda-2r)/(1-r)}d\epsilon.$$

Thus, if we take the meson spectrum as $\epsilon^{-2.8}d\epsilon$ and take $r=\frac{1}{3}$, the primary spectrum implied is $E_0^{-2.5}dE_0$ in the relatively high energy region for which these considerations are valid, E_0 greater than around 10-in. ev. For lower energies the multiplicity changes more rapidly, and if the primary spectrum predicted above persists below 10^{11} volts, we should expect the high altitude meson spectrum to deviate from an $\epsilon^{-2.8}$ law, the deviation being toward increased numbers of slow mesons. Of course the decay of the slower mesons and the magnetic effects on the primaries will influence these results.

C. Positive Excess

If we suppose that all final charge states of the nucleons are equally probable, then the fact that the primaries are positively charged causes an excess of positive mesons to be produced. If the number of charged mesons is N_c, then the average number of positive mesons will be $\frac{1}{4}(2N_c+1)$ and of negative $\frac{1}{4}(2N_c-1)$ so that the positive excess is $2/(2N_c-1)\sim 1/N_c$. This is both energy dependent and depends upon the absolute number of mesons produced. Taking, as above, $\bar{N}\sim 2E_0^{1/3}$ and $N_c\sim\frac{2}{3}\bar{N}\sim 1.3E_0^{1/3}$, we have

$$\text{Pos. excess}\sim 0.8E_0^{-1/3}\sim\epsilon^{-1/2}, \tag{57}$$

since $\epsilon_{\text{Av}}=E_0/N\cong\frac{1}{2}E_0^{2/3}$. This gives the positive excess as a function of meson energy at the point of production, and around 2×10^9 ev must be added to energies at sea level to extrapolate back to the top of the atmosphere. For energies of a few Bev at sea level, this corresponds to an excess of between 15 and 20 percent, which is in accord with the experiments. No data are yet available on the variation of the excess with meson energy.

D. Range of Fields and Distances Involved

Integrating the distribution (51) over angles, we obtain for the energy distribution in the laboratory system, which is the initial rest system for the air nucleon,

$$P(k_0)dk_0\cong\frac{N}{E_0}\exp\left[-N\left(\frac{k_0}{E_0}+\frac{1}{2Mk_0}\right)\right]dk_0. \tag{58}$$

Thus the energy distribution is uniform from around $k_0\sim N/2M$ to $k_0\sim E_0/N$, and the average energy is indeed E_0/N. Thus we are exploring the meson field to distances of order N/E_0 which, for $N=\bar{N}\sim 2E_0^{1/3}$ is $2E_0^{-2/3}$. Thus a cut-off at a few times the proton Compton wave-length, as is suggested by magnetic moment, nuclear force, and self-energy consideration, would modify our results in the entire cosmic-ray range. The proton Compton wave-length itself becomes important at around 10^{10} ev, and at 10^{15} volt we are exploring the meson field to distances of about 1/1000 of $\hbar/Mc$.